PHYSICAL REVIEW E VOLUME 57, NUMBER 3 MARCH 1998

Spatially localized unstable periodic orbits of a high-dimensional chaotic system

Scott M. Zoldi*,† and Henry S. Greenside*
Department of Physics, Duke University, Durham, North Carolina 27708-0305
(Received 14 April 1997)

Using an innovative damped-Newton method, we report the calculation and analysis of many distinct unstable periodic orbits (UPOs) for a high-fractal-dimension ($D=8.8$) extensively chaotic solution of a partial differential equation. A majority of the UPOs turn out to be spatially localized in that time dependence occurs only on portions of the spatial domain. With a escape-time weighting of 127 UPOs, the attractor's fractal dimension can be estimated with a relative error of 2%. Statistical errors are found to decrease as $1/\sqrt{N}$ as the number N of known UPOs increases. [S1063-651X(98)50703-9]

PACS number(s): 05.45.+b, 05.70.Ln, 47.27.Cn, 82.40.Bj

Over the last ten years, there has been a blossoming of research concerning the set of unstable periodic orbits (UPOs) associated with a chaotic attractor in phase space. Researchers have shown that knowledge of the short-period UPOs can often be used to estimate dynamical invariants of a chaotic attractor such as its fractal dimension and Lyapunov exponents [1] and to improve forecasting of time series generated by the attractor [2]. In some cases, the shorter-period UPOs can be extracted from time series and these can then be used to characterize experimental chaos [3]. For engineering applications in which chaos is undesirable, researchers have discovered algorithms that can convert chaotic to periodic behavior by stabilizing a given UPO through weak parameter modulations [4].

These many achievements are based on mathematical assumptions that restrict their applicability to low-dimensional systems. Theory that expresses the natural measure of an attractor in terms of the set of UPOs requires the assumption of hyperbolicity [5], which fails for most dynamical systems because of tangencies of stable and unstable manifolds [1] and because of unstable dimension variability [6]. The powerful cycle expansion method that expresses averages in terms of a moderate number of UPOs [1] is practical only if a symbolic dynamics (unique labeling of each UPO [1]) is explicitly known and it is widely believed that most dynamical systems lack a symbolic dynamics. Even if a system is hyperbolic and has an explicit symbolic dynamics, it is still nontrivial to calculate a relatively complete set of UPOs from specified equations or from measured time series to apply a cycle expansion. In the absence of hyperbolicity and of a symbolic dynamics, it is not known how to weight a given set of UPOs so as to approximate a given statistical average in a high-dimensional regime.

A consequence of these restrictions is that extremely little is known about the relation of UPOs to high-dimensional chaotic attractors such as those associated with large or strongly driven nonequilibrium systems [7,8]. An improved understanding of the spatial structure of UPOs, of the distribution of their periods T, and of their stabilities will also likely aid the development of high-dimensional spatiotemporal control algorithms by suggesting the number and location of control points for a particular UPO and for a particular system parameter that is varied.

In this Rapid Commuication, we take a significant step towards understanding the relation of the set of UPOs to high-dimensional spatiotemporal chaos by reporting the calculation and analysis of many (over 100) distinct UPOs for a high-fractal-dimension ($D=8.8$) driven-dissipative partial differential equation (PDE) [9]. This calculation represents three advances. One is numerical, that a simple modification of a Newton algorithm by the addition of damping [10] greatly increases the likelihood of convergence and so makes practical the computation of many UPOs. The second achievement is several discoveries in nonequilibrium physics, e.g., that most of the UPOs turn out to be spatially localized as discussed below and that about 100 UPOs are already sufficient to estimate the fractal dimension of a high-dimensional chaos to two significant digits. The third achievement is an empirical discovery, that a weighting of UPOs based on escape times can approximate several statistical averages accurately. These results suggest that a statistical theory of high-dimensional attractors in terms of UPOs might be possible even in the absence of hyperbolicity or of a symbolic dynamics.

Our calculations were carried out for one of the simplest models of spatiotemporal chaos, the one-dimensional Kuramoto-Sivashinsky (KS) equation [11]

$$\partial_t u = -\partial_x^2 u - \partial_x^4 u - u\,\partial_x u, \quad x\in[0,L], \tag{1}$$

where the field $u(t,x)$ exists on an interval of length L and satisfies rigid boundary conditions $u=\partial_x u=0$. For system sizes $L\geqslant 50$, Manneville has shown that typical initial conditions evolve towards a chaotic attractor that is extensive in that the Lyapunov fractal dimension D increases linearly with L [11]. In our calculations, we chose a fixed length $L=50$ and spatial resolution $\Delta x=0.5$ for which the Lyapunov fractal dimension was $D=8.8$ and there were 4 positive, 1 zero, and 94 negative Lyapunov exponents [12]. The system size $L=50$ was just large enough to be in the extensively chaotic regime and yet small enough that the numerical calculations were manageable with available resources and algorithms.

*Also at Center for Nonlinear and Complex Systems, Duke University, Durham, NC 27708.
†Electronic address: zoldi@phy.duke.edu

The problem of calculating UPOs of Eq. (1) can be posed as a set of nonlinear equations which specify that an orbit $\mathbf{U}(t)$ starting at a certain point $\mathbf{U}_0$ in the numerical phase space will close on itself after a period T [13]. By introducing a single vector of unknowns $\mathbf{X}=(T,\mathbf{U}_0)$, these nonlinear equations can be written abstractly as $\mathbf{0}=\mathbf{F}(\mathbf{X})=\mathbf{U}(T)-\mathbf{U}_0$ where the specific form of the vector function $\mathbf{F}(\mathbf{X})$ will depend on how Eq. (1) is discretized (we used second-order finite differences to approximate the spatial derivatives on a uniform mesh with spacing Δx). A standard way to solve these nonlinear equations is then a Newton method [13,10], in which a current estimate $\mathbf{X}$ of the unknowns is improved by adding a correction $\delta\mathbf{X}=-\mathbf{J}^{-1}(\mathbf{X})\mathbf{F}(\mathbf{X})$, where $\mathbf{J}=\partial\mathbf{F}/\partial\mathbf{X}$ is the Jacobian matrix. The iteration $\mathbf{X}\leftarrow\mathbf{X}+\delta\mathbf{X}$ is repeated until the magnitudes of the correction $\|\delta\mathbf{X}\|$ and of the residual $\|\mathbf{F}(\mathbf{X})\|$ are sufficiently small [14].

For high-dimensional Newton methods, it is essential to have a good starting guess since Newton methods are guaranteed to converge only locally and often diverge for initial values that are not close to a solution. We failed to find good initial guesses $\mathbf{X}_0=(T_0,\mathbf{U}_0)$ by searching for approximate recurrences [2,3] of chaotic time series $\mathbf{U}_i=\mathbf{U}(i\Delta t)$ in the 99-dimensional numerical phase space of Eq. (1). For example, for $L=50$, no approximate recurrences were found for a large integration time of 10^8 time units within a ball of rather large radius 0.1, $\|\mathbf{U}(T+t)-\mathbf{U}(t)\|_\infty<0.1$ [15].

Since no approximate recurrence was close to a UPO of Eq. (1), we then tried to choose an initial guess $\mathbf{X}_0$ by assigning a positive random number T for the period and choosing an initial vector $\mathbf{U}_0$ from a point on the numerically calculated chaotic attractor. This also failed to converge unless *damping* was introduced [10], in which only a fraction $\alpha\in(0,1]$ of a Newton correction was added to update the unknowns, $\mathbf{X}\leftarrow\mathbf{X}+\alpha\delta\mathbf{X}$. Damping is a widely used strategy in many numerical problems, in which convergence of a Newton method is improved by solving a related sequence of one-dimensional minimization problems [10]. Using a particular damping algorithm known as the Armijo rule [10], we obtained convergence [14] for 5% of all initial guesses tried. A deeper insight into what determines this success rate is unlikely at this time, since it would require an understanding of the basins of attraction associated with the high-dimensional Newton map of the discretized KS equation.

We now discuss the properties of the UPOs calculated with the above numerical methods. Using the damped-Newton method discussed above, 262 UPOs were found out of 5000 initial guesses $\mathbf{X}_0=(T_0,\mathbf{U}_0)$ of which 127 UPOs were distinct [16]. UPOs with periods shorter than 8 were not found while the Armijo-Newton algorithm failed to converge for UPOs with periods larger than 42. As shown qualitatively in Fig. 1 and more quantitatively in Fig. 2, a surprising feature of the calculated UPOs is that a majority is spatially localized in that the time variation is substantial only in isolated portions of the domain. The spatially localized dynamics of these UPOs suggests a mechanism by which a large chaotic system becomes extensive, acting as statistically independent subsystems [8]. The majority of the UPOs are not flip-symmetric but occur in pairs that preserve the inversion symmetry of the attractor. The corresponding mean and variance patterns [16] are given in Figs. 2(a)–2(f). In Figs. 2(a), 2(c), and 2(e), the mean pattern is nonzero

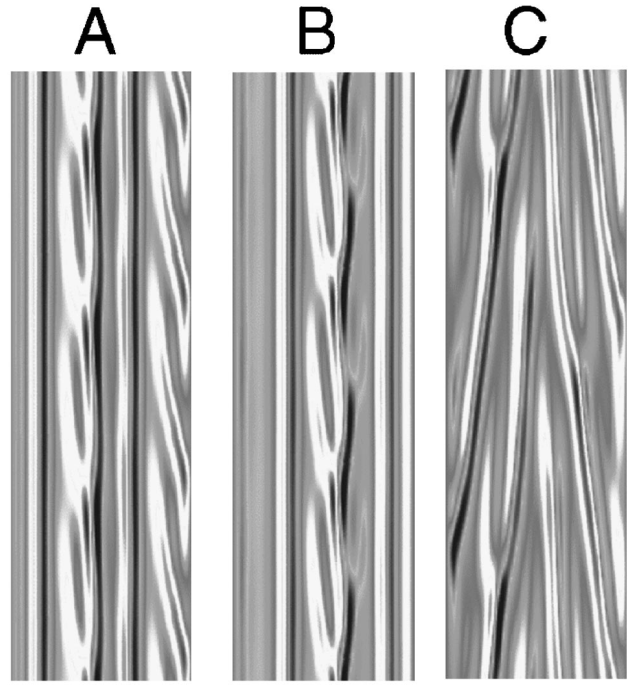

FIG. 1. Density plots of three representative UPOs $u(t,x)$ of Eq. (1) in a spatial domain of length $L=50$. The horizontal axis is space and the vertical axis spans a time interval of 35 time units. (a) A UPO of period $T=9.9$ with dynamics localized near the right boundary; (b) a UPO of period $T=10.7$ with dynamics localized in the interior of the interval; (c) a nonlocalized UPO of period $T=23.4$. The greyscales represent amplitude variations between 3 and -3.

throughout the domain, which holds also for the other 124 UPOs. Figures 2(b) and 2(d) indicate more clearly the localization of the dynamics, which is evidently uncorrelated with the mean pattern. The variance decreases three to five orders of magnitude outside the regions of substantial variation.

Using the explicit knowledge of the space-time evolution of our computed 127 distinct UPOs Eq. (1), we explored to what extent important quantities such as a mean spatial pattern and a fractal dimension can be approximated in terms of UPOs. Time-averaged patterns of spatiotemporal chaos have been recently found experimentally and are not yet understood [17]. If the dynamics was hyperbolic and ergodic, and if the UPOs could be ordered by symbolic length, then a trace formula could be used to approximate averages [1]. Empirically, no such ordering could be found rendering the trace formula inapplicable (see below).

Because of our inability to order UPOs by symbolic length, we developed an escape-time weighting $w=1/(\Sigma_+\lambda)$ for the contribution of each UPO in an average, based on the local instability $\Sigma_+\lambda$ of a UPO, given by the sum over all positive transverse Lyapunov exponents $\lambda=\log(|m|)/T$, where m is a Floquet multiplier of a UPO of period T. The number of positive Lyapunov exponents for each UPO varied between 3 and 8 and the largest Lyapunov exponent of the UPOs varied between 0.02 and 0.34. The escape-time weighting w reflects the fraction of time that a chaotic orbit spends in the vicinity of a particular UPO. Using this new-

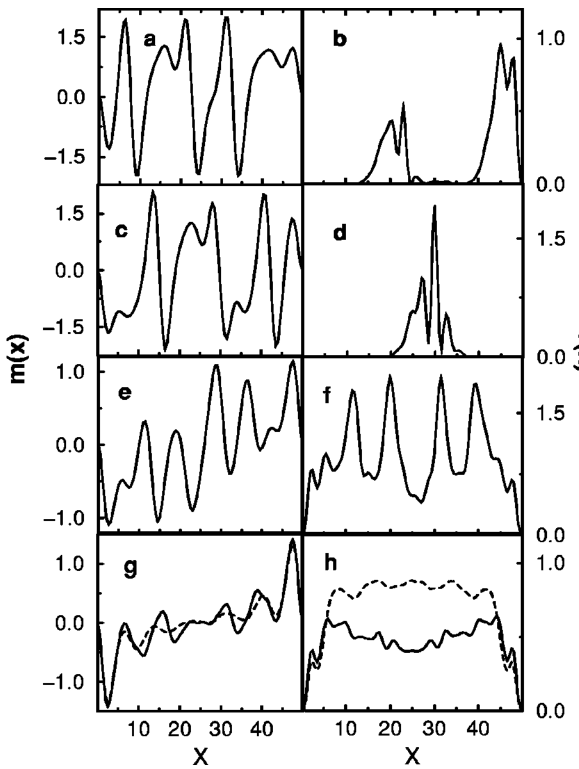

FIG. 2. Time-averaged mean patterns $m(x)=\langle u(t,x)\rangle$ and variance patterns $v(x)=\langle (u(t,x)-m(x))^2\rangle$ for the three representative UPOs of Fig. 1. (a) and (b): for the UPO with dynamics localized near a boundary. (c) and (d): For the UPO with dynamics localized away from boundaries. (e) and (f): For the extended UPO. (g) and (h): Mean and variance patterns (solid lines) averaged over all 127 distinct UPOs using the escape-time weighting. For comparison, the dashed lines give the corresponding mean and variance patterns obtained from an integration of a chaotic solution over 10^6 time units.

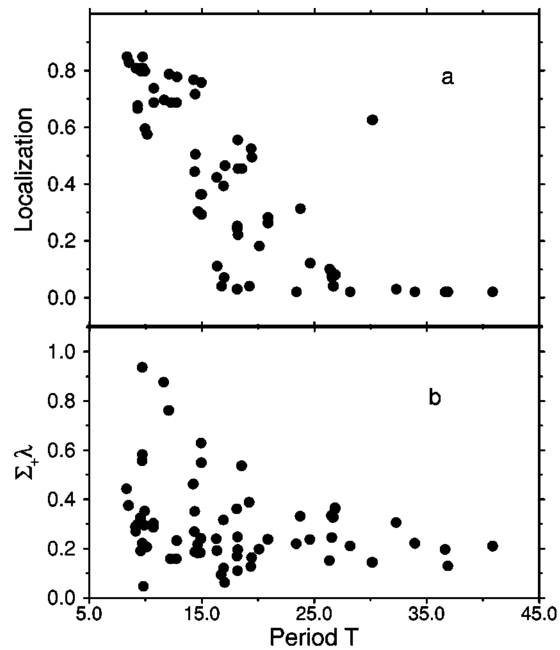

FIG. 3. (a) Localization [fraction of the spatial domain that has variance $v(x)$ below 0.05] vs period T of 127 distinct UPOs calculated for the KS equation [Eq. (1)] in an extensively chaotic regime with system size $L=50$. (b) Degree of instability as measured by the sum $\Sigma_+\lambda$ of positive transverse Lyapunov exponents vs the period T for all 127 UPOs.

weighting and averaging over the $N=127$ time-averaged patterns of each individual UPO $u_i(t,x)$ with period T_i we found rather good agreement [solid curve in Fig. 2(g)] with the mean pattern $m(x)=\langle u(t,x)\rangle$ obtained by a direct average of the chaotic field $u(t,x)$ over 10^6 time units [dashed curve in Fig. 2(g)]. The relative error in the infinity norm between the two averages is 24%, and is substantially better near the boundaries. The trace formula average without symbolic ordering of UPOs is unable to reproduce even the qualitative features of the mean pattern. An escape-time average of the 127 variance patterns [solid line in Fig. 2(h)] does not agree as well with the variance pattern (the relative error is 46%) but there is still a qualitative similarity.

Figure 3 shows how the extent of localization and instability depends on the period T. The degree of localization was defined as the fraction of the interval $[0,L]$ for which the variance $v(x)$ was smaller than 0.05 (the results were not sensitive to the choice of this cutoff). Figure 3(a) shows the localization versus the period T for all 127 UPOs. Although there is scatter in the points, we see that shorter period UPOs tend to be more strongly localized, that there can be many UPOs of approximately the same period (say $T=14$), and that the UPOs can vary substantially in their localization. In Fig. 3(b), we summarize the instability of all 127 UPOs as a function of their period T and find that there is a trend with smaller period UPOs being more unstable.

The same instability versus period trend does not hold in the classic low-dimensional Lorenz equations for which we have used the same damped-Newton method to calculate over 700 UPOs associated with the chaotic attractor. We speculate that the decay in instability of UPOs versus period for the KS equation [Fig. 3(b)] is due to the fact that we are not able to compute the most unstable high-period UPOs. Further, without the assumption of a symbolic dynamics and using the above escape-time weighting in the Lorenz equations, we were able to estimate the fractal dimension to an accuracy close to that obtained by a cycle expansion, giving further verification of the escape-time weighting.

Using the data in Fig. 3(b), the fractal dimension D of the chaotic attractor was estimated as follows. First, a fractal dimension was associated with each UPO by expressing the Kaplan-Yorke formula [11] in terms of its transverse Lyapunov exponents; we found dimensions ranging from 6 to 12 for the 127 UPOs. An escape-time weighting of the 127 dimensions then gave an estimate $D=9.0\pm 0.1$ for the fractal dimension of the attractor, a relative error of 2% compared to the Lyapunov fractal dimension $D=8.8\pm 0.1$ calculated directly from the Lyapunov exponents of the spatiotemporal chaotic solution of Eq. (1) [11]. The convergence of the escape time estimate to the Lyapunov dimension is statistical in that the error decreases approximately as $1/\sqrt{N}$, where N is the number of UPOs contributing to the weighted sum. Other previously published weightings of the UPOs were tried [2,5] but were found not to give results as accurate as our escape-time weighting, with relative errors larger than 10%.

In conclusion, we have used a damped-Newton algorithm to calculate many UPOs associated with a high-fractal-dimension chaotic solution of a PDE in a large spatial domain, Eq. (1). An important numerical insight was the use of damping to increase the likelihood of convergence of an otherwise straightforward Newton method. Damping was especially important, since no close recurrences could be found even over $T=10^8$ time units. The 127 distinct UPOs found were also used to predict successfully the qualitative features of the time-averaged mean pattern and the variance of the chaotic attractor. We could estimate the fractal dimension of the attractor to be 9.0 ± 0.1 compared to the actual value of 8.8 ± 0.1. The UPOs are typically localized in space which suggests a new way to think about the dynamically independent subsystems associated with extensive chaos. The localization also has important implications for the control of large chaotic systems using distributed sets of control points. [4]

We thank L. Bunimovich, S. Newhouse, and M. Strain for useful discussions. This work was supported by the Computational Science Fellowship Program of the Office of Scientific Computing in the Department of Energy, by NSF Grant Nos. NSF-DMS-93-07893 and NSF-CDA-92123483-04, and by DOE Grant No. DOE-DE-FG05-94ER25214.

[1] R. Artuso, E. Aurell, and P. Cvitanović, Nonlinearity **3**, 325 (1990).

[2] K. Pawelzik and H. G. Schuster, Phys. Rev. A **43**, 1808 (1991).

[3] D. P. Lathrop and E. J. Kostelich, Phys. Rev. A **40**, 4028 (1989); R. Badii *et al.*, Rev. Mod. Phys. **66**, 1389 (1994).

[4] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); K. Pyragas, Phys. Lett. A **170**, 421 (1992); G. Chen and X. Dong, Int. J. Bifurcation Chaos **3**, 1363 (1993).

[5] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).

[6] E. J. Kostelich, I. Kan, C. Grebogi, E. Ott, and J. Yorke, Physica D **109**, 81 (1997).

[7] A. Politi and A. Torcini, Phys. Rev. Lett. **69**, 3421 (1992); Chaos **2**, 293 (1992).

[8] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[9] In previous work, Politi *et al.* [7] have studied UPOs of a one-dimensional extensively chaotic system with *discrete* time, consisting of up to six weakly coupled Hénon maps. In contrast, our analysis concerns a continuous-time strongly-coupled system with an emphasis on the spatial structure of the UPOs. Also, Kevrekidis and collaborators [A. Bangia *et al.*, SIAM J. Sci. Comp. **18**, 775 (1997); Jolly *et al.* Physica D **44**, 38 (1990)] have computed periodic orbits from partial differential equations in a low-fractal-dimension or stable regime and studied bifurcations of these solutions. Our emphasis is again different; we start in a high-fractal-dimension regime where we do not have weakly unstable solutions to apply continuation.

[10] J. E. Dennis and R. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations* (Prentice-Hall, Englewood Cliffs, NJ, 1983).

[11] P. Manneville, in *Macroscopic Modeling of Turbulent Flows*, Vol. 230 of Lecture Notes in Physics, edited by O. Pironneau (Springer-Verlag, New York, 1985), pp. 319–326.

[12] Recently, F. Christiansen, P. Cvitanović and V. Putkaradze Nonlinearity **10**, 55 (1997) also studied the KS equation to investigate how a chaotic attractor may be related to the UPOs whose closure defines that attractor. Their emphasis and results differ from ours in that these authors considered a smaller *periodic* domain for which the chaotic dynamics is low-dimensional (we estimate $D<2.1$ from their data) and for which the set of UPOs has an explicit symbolic dynamics.

[13] T. S. Parker and L. O. Chua, *Practical Numerical Algorithms for Chaotic Systems* (Springer-Verlag, New York, 1989).

[14] We used the convergence criterion $\|\mathbf{U}(T)-\mathbf{U}_0\|_\infty<10^{-3}\|\mathbf{U}_0\|_\infty$ for the residual and $\|\delta\mathbf{X}\|_\infty<10^{-3}\|\mathbf{X}_0\|_\infty$ for the correction, where the infinity norm $\|\mathbf{X}\|_\infty=\max_i|X_i|$ and where the vector $\mathbf{U}(T)$ is the numerical trajectory evaluated at time $t=T$ that passes through the initial value $\mathbf{U}_0$ at time $t=0$.

[15] Our inability to find approximate recurrences from numerical output $\mathbf{U}(t)$ suggests that some of the widely used control methods [4] will not succeed in stabilizing UPOs since they depend on knowing an approximate recurrence as input to the algorithm.

[16] UPOs were considered distinct if their time-averaged pattern $m(x)=\langle u(t,x)\rangle$ and variance $v(x)=\langle (u(t,x)-m(x))^2\rangle$ averaged over one period differed by at least 0.01 and if their periods differed by at least 0.01.

[17] B. J. Gluckman, C. B. Arnold, and J. P. Gollub, Phys. Rev. E **51**, 1128 (1995).